Contents

2 Quantum Fields

Exercise 2.1

The partition function is

$$Z = \sum_n \mathrm{e}^{-\beta\hbar\omega(n+\frac{1}{2})} = \mathrm{e}^{-\beta\hbar\omega/2}\frac{1}{1-\mathrm{e}^{-\beta\hbar\omega}}\,. \tag{2.1.1}$$

Let $x = \beta\hbar\omega$,

$$Z = \mathrm{e}^{-x/2}(1-\mathrm{e}^{-x})^{-1}\,. \tag{2.1.2}$$

The internal energy is

$$\begin{aligned} E &= N_A\left(-\frac{\partial \ln Z}{\partial \beta}\right) \\ &= -N_A\hbar\omega\frac{1}{Z}\frac{\partial Z}{\partial x} \\ &= \frac{1}{2}N_A\hbar\omega\left(1+\frac{\mathrm{e}^{-x/2}}{\sinh\frac{x}{2}}\right)\,. \end{aligned} \tag{2.1.3}$$

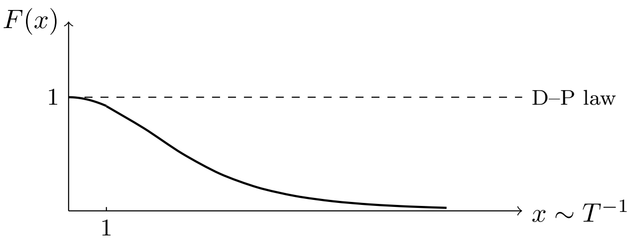
The heat capacity is

$$\begin{aligned} C_V &= \frac{\partial E}{\partial T} = \frac{\mathrm{d}x}{\mathrm{d}T}\frac{\partial E}{\partial x} \\ &= \frac{1}{4}Rx^2\frac{1}{\sinh^2(\frac{x}{2})} \\ &= RF(x) \end{aligned} \tag{2.1.4}$$

where

$$F(x) = \left(\frac{x/2}{\sinh(x/2)}\right)^2 \to 1 \tag{2.1.5}$$

in the $x \to 0$ limit ($k_BT \gg \hbar\omega$), where it recovers the Dulong–Petit law.



Exercise 2.2

$$\begin{aligned} \sum_j \langle q_m|j\rangle\langle j|q_n\rangle &= \frac{1}{N_S}\sum_{j=0}^{N_S-1}\mathrm{e}^{\mathrm{i}(q_n-q_m)ja} \\ &= \frac{1}{N_S}\sum_{j=0}^{N_s-1}\mathrm{e}^{\mathrm{i}(n-m)\frac{2\pi}{N_S}j}\,. \end{aligned} \tag{2.2.1}$$

If $n \neq m$, then[1]

$$\begin{aligned}
\text{LHS} &= \frac{1}{N_S} \frac{\mathrm{e}^{\mathrm{i}2\pi(n-m)} - 1}{\mathrm{e}^{\mathrm{i}2\pi(n-m)/N_S} - 1} \\
&= \frac{1}{N_S} \frac{\sin(\pi(n-m))}{\sin\left(\frac{\pi}{N_S}(n-m)\right)} \frac{\mathrm{e}^{-\mathrm{i}\pi(n-m)/N_S}}{\mathrm{e}^{-\mathrm{i}\pi(n-m)}} \\
&= \frac{1}{N_S} \frac{\sin(\pi(n-m))}{\sin\left(\frac{\pi}{N_S}(n-m)\right)} (-1)^{\frac{N_S-1}{N_S}(n-m)} \\
&= 0\,.
\end{aligned} \tag{2.2.2}$$

If $n = m$, then

$$\begin{aligned}
\text{LHS} &= \frac{1}{N_S} \sum_{j=0}^{N_S-1} 1 \\
&= 1\,.
\end{aligned} \tag{2.2.3}$$

Therefore,

$$\langle q_m | q_n \rangle = \delta_{mn}\,. \tag{2.2.4}$$

Exercise 2.3

Have

$$\hat{x}(0) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \tag{2.3.1}$$

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}\mathrm{e}^{\mathrm{i}\omega t} + \hat{a}^\dagger \mathrm{e}^{-\mathrm{i}\omega t})\,. \tag{2.3.2}$$

The symmetrised correlation function can be expanded as

$$\begin{aligned}
S_\beta(t) &= \frac{1}{2} \langle \hat{x}(t)\hat{x}(0) + \hat{x}(0)\hat{x}(t) \rangle \\
&= \frac{\hbar}{4m\omega} \left[2\mathrm{e}^{\mathrm{i}\omega t} \langle \hat{a}\hat{a} \rangle + 2\mathrm{e}^{-\mathrm{i}\omega t} \left\langle \hat{a}^\dagger \hat{a}^\dagger \right\rangle + 2\cos\omega t \left(\left\langle \hat{a}\hat{a}^\dagger \right\rangle + \left\langle \hat{a}^\dagger \hat{a} \right\rangle \right) \right]\,.
\end{aligned} \tag{2.3.3}$$

At (inverse) temperature β,

$$\begin{cases}
\left\langle \hat{a}^\dagger \hat{a} \right\rangle = \langle \hat{n} \rangle \\
\left\langle \hat{a}\hat{a}^\dagger \right\rangle = \left\langle \hat{a}^\dagger \hat{a} + 1 \right\rangle = \langle \hat{a} \rangle + 1 \\
\langle \hat{a}\hat{a} \rangle = \left\langle \hat{a}^\dagger \hat{a}^\dagger \right\rangle = 0\,,
\end{cases} \tag{2.3.4}$$

where

$$\langle \hat{n} \rangle = n(\beta) = \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}\,. \tag{2.3.5}$$

Substituting in (2.3.3) gives

$$S_\beta(t) = \frac{\hbar}{2m\omega} \cos(\omega t) [\underbrace{2n(\beta)}_{\text{finite-}T\text{ excitation}} + \underbrace{1}_{\text{zero-point fluctuation}}]\,. \tag{2.3.6}$$

[1] I believe the stated result is missing the red N_S^{-1}.

Exercise 2.4

(a)

$$\begin{aligned}[\hat{b},\hat{b}] &= u^2[\hat{a},\hat{a}] + uv[\hat{a},\hat{a}^\dagger] + uv[\hat{a}^\dagger,\hat{a}] + v^2[\hat{a}^\dagger,\hat{a}^\dagger] \\ &= 0 \end{aligned} \tag{2.4.1}$$

$$\begin{aligned}[\hat{b}^\dagger,\hat{b}^\dagger] &= u^2[\hat{a}^\dagger,\hat{a}^\dagger] + uv[\hat{a}^\dagger,\hat{a}] + uv[\hat{a},\hat{a}^\dagger] + v^2[\hat{a},\hat{a}] \\ &= 0 \end{aligned} \tag{2.4.2}$$

$$\begin{aligned}[\hat{b},\hat{b}^\dagger] &= u^2[\hat{a},\hat{a}^\dagger] + uv[\hat{a},\hat{a}] + uv[\hat{a}^\dagger,\hat{a}^\dagger] + v^2[\hat{a}^\dagger,\hat{a}] \\ &= u^2 - v^2\,. \end{aligned} \tag{2.4.3}$$

The first two canonical commutation relations of $\hat{b}$ are guaranteed by the canonical commutation relations of $\hat{a}$, while the last one satisfies $[\hat{b},\hat{b}^\dagger] = 1$ if and only if $u^2 - v^2 = 1$.

This means that it is convenient to parameterise u, v as $u = \cosh\theta$, $v = \sinh\theta$.

(b) Given $u^2 - v^2 = 1$, the inverse transform is

$$\hat{a} = u\hat{b} - v\hat{b}^\dagger\hat{a}^\dagger = u\hat{b}^\dagger - v\hat{b}\,. \tag{2.4.4}$$

Then we can expand

$$\begin{aligned}\hat{a}^\dagger\hat{a} + \frac{1}{2} &= u^2\hat{b}^\dagger\hat{b} - uv\hat{b}^\dagger\hat{b}^\dagger - uv\hat{b}\hat{b} + v^2\hat{b}^\dagger\hat{b}^\dagger + \frac{1}{2} \\ &= (u^2+v^2)\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) - uv(\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{b}) \end{aligned} \tag{2.4.5}$$

and similarly

$$\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger = (u^2+v^2)(\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger) - 4uv\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) \tag{2.4.6}$$

using $u^2 - v^2 = 1$. Therefore,

$$\hat{H} = [\omega(u^2+v^2) - 2\Delta uv]\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \left[\frac{\Delta}{2}(u^2+v^2) - \omega uv\right](\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{b})\,. \tag{2.4.7}$$

For $\hat{H}$ to be diagonalisable, there must be some θ such that for $u = \cosh\theta$, $v = \sinh\theta$, the latter term vanish,

$$\Delta(u^2+v^2) = 2\omega uv\,. \tag{2.4.8}$$

In θ, this is

$$\Delta\cosh 2\theta = \omega\sinh 2\theta\,, \tag{2.4.9}$$

so

$$\tanh 2\theta = \frac{\Delta}{\omega}\,. \tag{2.4.10}$$

Therefore, we must have

$$|\Delta| < \omega \tag{2.4.11}$$

for $\hat{H}$ to be diagonalisable.

The new frequency is

$$\begin{aligned}\tilde{\omega} &= \omega(u^2+v^2) - 2\Delta uv \\ &= \omega\cosh 2\theta - \Delta\sinh 2\theta\,. \end{aligned} \tag{2.4.12}$$

Given $\tanh 2\theta = \frac{\Delta}{\omega}$, we have

$$\begin{cases} \sinh 2\theta = \frac{\Delta}{\sqrt{\omega^2-\Delta^2}} \\ \cosh 2\theta = \frac{\omega}{\sqrt{\omega^2-\Delta^2}}\,, \end{cases} \tag{2.4.13}$$

so

$$\tilde{\omega} = \sqrt{\omega^2 - \Delta^2}\,. \tag{2.4.14}$$

Since

$$\begin{cases} u^2 + v^2 = \cosh 2\theta = \frac{\omega}{\tilde{\omega}} \\ u^2 - v^2 = 1\,, \end{cases} \tag{2.4.15}$$

we have

$$\begin{cases} u^2 = \frac{1}{2}\left(\frac{\omega}{\tilde{\omega}} + 1\right) \\ v^2 = \frac{1}{2}\left(\frac{\omega}{\tilde{\omega}} - 1\right)\,. \end{cases} \tag{2.4.16}$$

We need to take care of the signs when taking the square roots.

$$\begin{cases} u = \sqrt{\frac{1}{2}\left(\frac{\omega}{\sqrt{\omega^2-\Delta^2}} + 1\right)} \\ v = \mathrm{sgn}(\Delta)\sqrt{\frac{1}{2}\left(\frac{\omega}{\sqrt{\omega^2-\Delta^2}} - 1\right)}\,. \end{cases} \tag{2.4.17}$$

(c) We have

$$\mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\alpha\hat{a}^\dagger\hat{a}^\dagger\right)^k\,. \tag{2.4.18}$$

The commutation relations are

$$[\hat{a}^\dagger, \mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger}] = 0 \tag{2.4.19}$$

and

$$\begin{aligned} [\hat{a}, \mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger}] &= \sum_{k=0}^{\infty} \frac{1}{k!}(-\alpha)^k \left[\hat{a}, (\hat{a}^\dagger)^{2k}\right] \\ &= \sum_{k=1}^{\infty} \frac{1}{k!}(-\alpha)^k \cdot 2k(\hat{a}^\dagger)^{2k-1}\,, \end{aligned} \tag{2.4.20}$$

where in evaluating $[\hat{a}, (\hat{a}^\dagger)^n]$, notice that when you move $\hat{a}$ to the right of a string of $\hat{a}^\dagger$'s, each time you move through an $\hat{a}^\dagger$, you get a $[\hat{a}, \hat{a}^\dagger] = 1$ multiplied with the other $(n-1)$ $\hat{a}^\dagger$'s.

Therefore,

$$\begin{aligned} \hat{b}\,|\tilde{0}\rangle &= (u\hat{a} + v\hat{a}^\dagger)\mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger}\,|0\rangle \\ &= \left[u\left(\mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger}\hat{a} + [\hat{a}, \mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger}]\right) + v\mathrm{e}^{-\alpha\hat{a}^\dagger\hat{a}^\dagger}\hat{a}^\dagger\right]|0\rangle \\ &= \left\{u\sum_{k=1}^{\infty}\frac{1}{k!}(-\alpha)^k 2k(\hat{a}^\dagger)^{2k-1} + v\sum_{k=0}^{\infty}\frac{1}{k!}(-\alpha)^k(\hat{a}^\dagger)^{2k+1}\right\}|0\rangle \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}(-\alpha)^k\left[-\frac{\alpha}{k+1}\cdot 2(k+1)u + v\right](\hat{a}^\dagger)^{2k+1}\,|0\rangle\,. \end{aligned} \tag{2.4.21}$$

For this to be 0, each term in the sum must be zero, and hence

$$\alpha = \frac{v}{2u}\,. \tag{2.4.22}$$

Alternatively: suppose

$$\left|\tilde{0}\right\rangle = \sum_{n=0}^{\infty} c_n \left|n\right\rangle \ . \tag{2.4.23}$$

Then

$$\begin{aligned}\hat{b}\left|\tilde{0}\right\rangle &= (u\hat{a} + v\hat{a}^\dagger)\sum_{n=0}^{\infty} c_n \left|n\right\rangle \\ &= u\sum_{n=1}^{\infty} c_n \sqrt{n}\left|n-1\right\rangle + v\sum_{n=0}^{\infty} c_n\sqrt{n+1}\left|n+1\right\rangle \\ &= uc_1\left|0\right\rangle + \sum_{n=1}^{\infty}(uc_{n+1}\sqrt{n+1} + vc_{n-1}\sqrt{n})\left|n\right\rangle \ .\end{aligned} \tag{2.4.24}$$

Each term must vanish, so

$$\begin{cases} uc_1 = 0 \\ uc_{n+1}\sqrt{n+1} + vc_{n-1}\sqrt{n} = 0 \quad \text{for } n \geq 1\,. \end{cases} \tag{2.4.25}$$

The first condition implies $c_1 = 0$, while the second condition gives the iterative formula for $n \geq 2$:

$$c_n = -\frac{v}{u}\sqrt{\frac{n-1}{n}}c_{n-2}\,. \tag{2.4.26}$$

This means that all the odd terms must vanish:

$$c_{2k+1} = 0 \tag{2.4.27}$$

for $k \in \mathbb{N}$, while the even coefficients are

$$c_{2k} = \left(-\frac{v}{u}\right)^k \sqrt{\frac{(2k-1)!!}{(2k)!!}}c_0\,. \tag{2.4.28}$$

Therefore,

$$\begin{aligned}\left|\tilde{0}\right\rangle &= \sum_{k=0}^{\infty} c_{2k}\left|2k\right\rangle \\ &= \sum_{k=0}^{2k}\left(-\frac{v}{u}\right)^k \sqrt{\frac{(2k-1)!!}{(2k)!!}}\frac{(\hat{a}^\dagger)^{2k}}{(2k)!}\left|0\right\rangle \\ &= \sum_{k=0}\left(-\frac{v}{u}\right)^k \frac{1}{(2k)!!}(\hat{a}^\dagger)^{2k}\left|0\right\rangle \\ &= \left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left[-\frac{v}{2u}(\hat{a}^\dagger)^2\right]^k\right\}\left|0\right\rangle \\ &= \exp\left(-\frac{v}{2u}\hat{a}^\dagger\hat{a}^\dagger\right)\left|0\right\rangle \ .\end{aligned} \tag{2.4.29}$$

Note that the new vacuum is not the old vacuum. It contains indefinite, even number of the original bosons.